Confluent singularity of the renormalised coupling constant

1983 J. Phys. A: Math. Gen. 16 3899

(http://iopscience.iop.org/0305-4470/16/16/028)

J. Phys. A: Math. Gen. 16 (1983) 3899–3904. Printed in Great Britain

Confluent singularity of the renormalised coupling constant

Mau-chung Chang† and J J Rehr‡

† Sering Physics Laboratory, Rutgers University, Piscataway, NJ 08854, USA
‡ Department of Physics, University of Washington, Seattle, WA 98195, USA

Received 31 May 1983

Abstract. The amplitude of the confluent singular term in the dimensionless, renormalised coupling constant $u \sim \xi^{-d}(\partial^2\chi/\partial h^2)/\chi^2 = u^*(1 + a_u^+ t^{\omega\nu} + \ldots)$ is analysed using both renormalised perturbation theory and high-temperature series expansions by means of ε-expansion and second-order differential approximants, respectively. Results for the critical amplitude ratio a_u^+/a_χ^+ are within uncertainties in agreement, consistent with the prediction of universality.

1. Introduction

Near the critical point in systems that exhibit second-order phase transitions, a thermodynamic quantity f_i behaves as

$$f_i(t) = A_i|t|^{-\lambda_i}(1 + a_i|t|^{\Delta} + \mathrm{O}(|t|^{2\Delta})). \tag{1}$$

Here $t = (T - T_c)/T_c$, Δ is the Wortis–Wegner (Wortis 1970, Wegner 1972) correction to scaling exponent, and a_i is the amplitude of the confluent singular term. It has been shown that the ratios a_i/a_j are universal quantities (Aharony and Ahlers 1980, Chang and Houghton 1980a, b, 1981, Chang 1980). They have been calculated to zeroth order in $\varepsilon = 4 - d$ by Aharony and Ahlers (1980) using the Nelson–Rudnick (1975) renormalisation group approach, and also (Chang and Houghton 1980a, b, 1981, Chang 1980) to second order in ε using the massless renormalised perturbation theory (RPT) (Brézin *et al* 1976, Amit 1978). Recently Bagnuls and Bervillier (1981) have calculated these ratios using higher-order RPT estimates (Baker *et al* 1976) at $d = 3$ in the massive theory (Brézin *et al* 1976, Amit 1978). Ferer *et al* (1977; Rogiers *et al* 1979), Nickel (1980) and Nickel and Rehr (1983) have also given the values of some a_i/a_j using the high-temperature series expansion method (HTSE). Some experimental values are also available (Ahlers 1980, Bourgon and Beysens 1981).

Recently there has been some interest in HTSE of the dimensionless renormalised coupling constant $u \sim \xi^{-d}(\partial^2\chi/\partial H^2)/\chi^2$ (ξ is the correlation length, χ is the susceptibility, and H is the magnetic field) for the hyperscaling problem (Rehr 1979, Baker and Kincaid 1979).

The purpose of this paper is to analyse this coupling constant u by taking the confluent singular term into consideration using both HTSE and RPT. Some amplitude ratios like a_u^+/a_χ^+, a_u^+/a_ξ^+ and $a_{M_2}^+/a_\chi^+$ (+ means $T > T_c$, M_2 is the second moment of the spin–spin correlation function defined as $\xi^2 \sim M_2/\chi$) are given. The results of both theories are in agreement with each other.

2. Renormalised perturbation theory

We first discuss briefly renormalised perturbation theory. We start from a Ginzburg–Landau effective Hamiltonian H, with

$$\beta H = \int \mathrm{d}^d x \, H(x),$$
$$H(x) = \tfrac{1}{2}[\nabla\phi(x)]^2 + \tfrac{1}{2}\mu^2\phi^2(x) + (\lambda/4!)[\phi^2(x)]^2. \tag{2}$$

Here $\phi(x)$ is a local n-component vector field. Expressions for amplitude of the confluent singular term a_i have been given explicitly in the massless and massive RPT in Chang and Houghton (1980a, b, 1981), Chang (1980) and Bagnuls and Bervillier (1981), respectively. The distinction between the two different renormalisation schemes is that in the massless theory, the renormalisation conditions are fixed at $T = T_c$, while the massive theory is renormalised at $T \neq T_c$.

The starting point here is the scaling equation for the vertex function $\Gamma_R^{(L,N)}$ in the massless theory (see Chang and Houghton 1980a, b, 1981, Chang 1980, Brézin *et al* 1976, Amit 1978):

$$\Gamma_R^{(L,N)}(p_1 \ldots p_2, q_1 \ldots q_N; t\, \Lambda, g, \kappa, M)$$
$$= Z_{\phi^2}^{L} Z_{\phi}^{N/2}\, \Gamma^{(L,N)}(p_1 \ldots p_2, q_1 \ldots q_N; \mu^2, \Lambda, \lambda, \phi) \tag{3}$$

where

$$\mu^2 = \mu_c^2 + Z_{\phi^2} t, \tag{4}$$
$$\bar{\phi} = Z_{\phi}^{1/2} M, \tag{5}$$
$$g = \kappa^{\varepsilon} u. \tag{6}$$

Here $\Gamma_R^{(L,N)}$, t, M and g are renormalised quantities; μ_c^2 is the value of μ^2 at $T = T_c$; κ is an arbitrary momentum scale; and Z_ϕ and Z_{ϕ^2} are the renormalisation constants which can be calculated from the renormalisation conditions. When u is close to its fixed point u^*, we find

$$\Gamma_R^{(L,N)}(p_1 \ldots p_2, q_1 \ldots q_N; t, M = 0, u)$$
$$= Y^N X^L \tilde{t}^{\,\nu[d - N(d-2+\eta)/2] - L}\, \Gamma_R^{(L,N)}(p\tilde{t}^{-\nu}; t = 1, M = 0, u^*, \kappa = 1)$$
$$\times \left[1 + (u - u^*)\tilde{t}^{\,\omega\nu}\left(\frac{\partial\Gamma_R^{(L,N)}(p\tilde{t}^{-\nu}; t = 1, M = 0, u, \kappa = 1)/\partial u|_{u^*}}{\Gamma_R^{(L,N)}(p\tilde{t}^{-\nu}; t = 1, M = 0, u^*, \kappa = 1)}\right.\right.$$
$$\left.\left. - \frac{\tfrac{1}{2}N[\partial\eta(u)/\partial u]|_{u^*} + \nu[d - \tfrac{1}{2}N(d-2+\eta)](\partial\nu^{-1}/\partial u)|_{u^*}}{\omega}\right)\right], \tag{7}$$

which identifies the correction to scaling amplitude a_i. All the renormalised quantities can easily be rewritten in terms of bare quantities using (3), (4) and (5).

In the massive theory the renormalised vertex function $\Gamma_R^{(L,N)}$ is also related to the bare one by

$$\Gamma_R^{(L,N)}(\{q\}; \{p\}; m, u) = Z_3^{N/2}(Z_4/Z_3)\Gamma^{(L,N)}(\{q\}; \{p\}; \mu^2, \lambda, \Lambda). \tag{8}$$

The renormalisation conditions are defined at $T \neq T_c$. We should note that both u's in (7) and (8) are the dimensionless renormalised coupling constants in the massless and massive theory respectively. They are two completely different quantities, although we still use the same conventional notation. In the massless theory u is temperature independent, while in (8) it is a function of temperature.

From the renormalisation conditions, it is easy to see that $\chi^{-1}=\Gamma^{(0,2)}(0;\mu^2,\lambda)=Z_3^{-1}m^2$ and

$$\xi^2=\frac{\partial\Gamma^{(0,2)}(p;\mu^2;\lambda)}{\partial p^2}\bigg/\Gamma^{(0,2)}(p;\mu^2,\lambda)\bigg|_{p=0}=m^{-2}.$$

Also using

$$\chi=\partial\bar{\phi}/\partial H=[\Gamma^{(0,2)}(\bar{\phi},\mu^2)]^{-1}=[\partial^2\Gamma(\bar{\phi},\mu^2)/\partial\bar{\phi}^2]^{-1}$$

($\Gamma(\bar{\phi},\mu^2)$ is the free energy, $\bar{\phi}$ is the magnetisation), we can simply use chain differentiation to establish that

$$\begin{aligned}\xi^{-d}(\partial^2\chi/\partial H^2)/\chi^2&=-\xi^{-d}\chi^2\Gamma^{(4)}(p=0)=-m^dm^{-4}Z_3^{+2}\Gamma^{(4)}(p=0)\\&=-m^{-\varepsilon}\Gamma_{\mathrm{R}}^{(0,4)}(p=0)=-u\qquad(T>T_c).\end{aligned}\tag{9}$$

This is exactly the dimensionless renormalised coupling constant in the massive theory.

3. Confluent amplitude in RPT

We now calculate a_u^+ in the massless theory. Then using the known values of a_χ^+, a_ξ^+ and $a_{M_2}^+=(2a_\xi^++a_\chi^+)$, we can immediately determine a_u^+/a_χ^+, a_u^+/a_ξ^+ and $a_{M_2}^+/a_\chi^+$. We shall also reproduce these results in the massive theory. This serves as a check that such universal quantities are independent of the renormalisation scheme.

From (9), we have

$$\begin{aligned}u&=\xi^{-d}\chi^2\Gamma^{(0,4)}(p=0)=\xi_{\mathrm{R}}^{-d}\chi_{\mathrm{R}}^2\Gamma_{\mathrm{R}}^{(0,4)}(p=0)\\&=Y^{4-4}\tilde{t}^{d\nu-2\gamma+\nu(d-(d-2+\eta)2)}\Gamma_{\mathrm{R}}^{-d}(t=1,u^*)\chi_{\mathrm{R}}^2(t=1,u=u^*)\\&\quad\times\Gamma_{\mathrm{R}}^{(0,4)}(p=0;t=1,u^*)[1+t^{\omega\nu}(-da_{\xi\mathrm{R}}^++2a_{\chi\mathrm{R}}^++a_{\Gamma_{\mathrm{R}}^{(0,4)}}^+)]\\&=\xi_{\mathrm{R}}^{-d}(t=1,u^*)\chi_{\mathrm{R}}^2(t=1,u=u^*)\Gamma_{\mathrm{R}}^{(0,4)}(p=0;t=1,u^*)\\&\quad\times[1+t^{\omega\nu}(-da_{\xi\mathrm{R}}^++2a_{\chi\mathrm{R}}^++a_{\Gamma_{\mathrm{R}}^{(0,4)}}^+)].\end{aligned}\tag{10}$$

Note that $\gamma=\nu(2-\eta)$ has been used. The Feynman diagrams of $\Gamma_{\mathrm{R}}^{(0,4)}(p=0;t=1,u)$ are shown in figures 1 and 2. Therefore after some algebra we have

$$\begin{aligned}a_{\Gamma_{\mathrm{R}}^{(0,4)}}^+&=(u-u^*)X^{\omega\nu}\bigg(\frac{\partial\Gamma_{\mathrm{R}}^{(0,4)}(p=0;t=1,u)/\partial u|_{u^*}}{\Gamma_{\mathrm{R}}^{(0,4)}(p=0;t=1,u)}\\&\qquad-\frac{2\partial\eta(u)/\partial u|_{u^*}+\nu(d-2(d-2+\eta))\partial\nu^{-1}/\partial u|_{u^*}}{\omega}\bigg)\\&=(u-u^*)X^{\omega\nu}\frac{n+8}{6\varepsilon}\left[1+\varepsilon\left(\frac{3}{2}+\frac{n+2}{2(n+8)}-\frac{11n+46}{(n+8)^2}\right)\right].\end{aligned}\tag{11}$$

Using the known values (Chang and Houghton 1980a, b, 1981, Chang 1980)

$$a_{\xi\mathrm{R}}^+=-(u-u^*)X^{\omega\nu}\frac{n+2}{12\varepsilon}\left(1+\varepsilon\frac{3n^2+50n+148}{2(n+8)^2}\right),\tag{12}$$

$$a_{\chi\mathrm{R}}^+=-(u-u^*)X^{\omega\nu}\frac{n+2}{6\varepsilon}\left(1+\varepsilon\frac{3(n^2+16n+44)}{2(n+8)^2}\right),\tag{13}$$

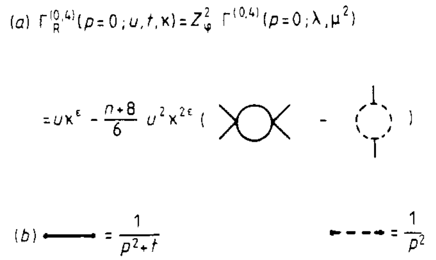

Figure 1. (*a*) Bare vetex function $\Gamma^{(0,4)}(p=0;\lambda,\mu^2)$ to first order of λ (bare coupling constant). (*b*) The Feyman propagator used in (*a*).

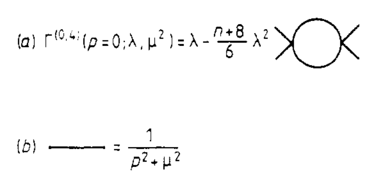

Figure 2. (*a*) Renormalised vertex function $\Gamma_R^{(0,4)}(p=0;u,t,\kappa)$ to first order of u in which κ is an arbitrary momentum scale. (*b*) The Feynman propagators used in (*a*).

we obtain

$$\begin{aligned} a_{uR}^{+} &= -(4-\varepsilon)a_{\xi R}^{+}+2a_{\chi R}^{+}+a_{\Gamma_R^{(0,4)}}^{+} \\ &= \frac{n+8}{6\varepsilon}\left[1+\varepsilon\left(\frac{3}{2}-\frac{9n+42}{(n+8)^2}\right)\right](u-u^*)X^{\omega\nu}. \end{aligned} \tag{14}$$

Then it follows that $(n=1, d=3)$

$$a_u^+/a_\chi^+ = -3(1+\varepsilon(-\tfrac{7}{27})) = -2.22, \tag{15}$$

$$a_u^+/a_\xi^+ = -6(1+\varepsilon(-\tfrac{10}{27})) = -3.78, \tag{16}$$

$$a_{M_2}^+/a_\chi^+ = 1+2a_\xi^+/a_\chi^+ = 1+(1+0.11\varepsilon) = 2.11. \tag{17}$$

We can check these results using the massive theory. Bagnuls and Bervillier (1981) have the formula

$$u = u^*+(Xt)^{\omega\nu}\left(\frac{A_1}{\lambda^{\varepsilon}}\right)^{\omega} = u^*\left[1+\frac{1}{u^*}(Xt)^{\omega\nu}\left(\frac{A_1}{\lambda^{\varepsilon}}\right)^{\omega}\right]. \tag{18}$$

Comparing (9) and (18),

$$\xi_R^{-d}(t=1,u^*)\chi_R^2(t=1,u=u^*)\Gamma_R^{(0,4)})p=0;t=1,u^*)$$

calculated in the massless theory indeed gives

$$\frac{6}{(n+8)}\left[1+\varepsilon\left(\frac{1}{2}+\frac{3(3n+14)}{(n+8)^2}\right)\right] = u^*$$

in the massive theory. We can take the ratio a_u^+/a_χ^+ and get

$$\frac{a_u^+}{a_\chi^+} = \frac{u^*}{\gamma_3^{(1)}/\omega+(2-\eta)[\nu/(\omega\nu+1)][(\gamma_4^{(1)}-\gamma_3^{(1)})/\omega-\gamma_3^{(1)}/(2-\eta)]}, \tag{19}$$

$$\gamma_3^{(1)}=\frac{\mathrm{d}\gamma_3(u)}{\mathrm{d}u}\bigg|_{u^*}=\frac{\mathrm{d}(\beta(u)(\mathrm{d}/\mathrm{d}u)\ln Z_3(u))}{\mathrm{d}u}\bigg|_{u^*}, \tag{20}$$

$$\gamma_4^{(1)}=\frac{\mathrm{d}\gamma_4(u)}{\mathrm{d}u}\bigg|_{u^*}=\frac{\mathrm{d}(\beta(u)(\mathrm{d}/\mathrm{d}u)\ln Z_4(u))}{\mathrm{d}u}\bigg|_{u^*}. \tag{21}$$

Using all the known values in the massive theory (Brézin *et al* 1973), equation (19) gives exactly the same result in (15).

4. Confluent amplitude in HTSE

The approach for estimating a_u^+ in HTSE is based on an analysis of the series $x(y)$ introduced by Nickel and Sharpe (1979) where x is the squared correlation length and $y=cu^{-2/d}$ (c is a constant chosen such ahat $x/y\to 1$, $y\to 0$). We follow the method outlined in Rehr (1979) based on second-order homogeneous differential approximants biased with a fixed value of y^*. Near y^*, one has

$$x(y)/y=A^+(1-y/y^*)^{-2/\omega}+B^+(1-y/y^*)^{-\gamma_2} \tag{22}$$

while near T_c, the squared correlation length diverges as

$$x=C^+t^{-2\nu} \tag{23}$$

where $t=(1-K/K_c)$, K_c being the critical value of J/KT. Combining (22) and (23) gives for $d=3$ the relation

$$a_u^+=\tfrac{3}{2}(A^+y^*/C^+)^{\omega/2}. \tag{24}$$

Since the precise value of y^* is uncertain, calculations were carried out at $y^*=$ 0.1602, 0.1608 and 0.1615 using series containing 12 and 13 terms. Since the work of Rehr (1979) and Nickel and Sharpe (1979), one additional term has been added to $x(y)$, owing to the availability of an additional term in the χ'' series (McKenzie 1979). It is found that the critical amplitudes are linearly correlated with the value of the leading exponent, with a slope $\delta A^+/\delta\omega\sim 0.1$. Also it is found that the results of various approximants for *both* 12 and 13 terms are most nearly consistent with $y^*\cong 0.1608$, somewhat higher than the value 0.1602 reported in Rehr (1979). Our results are summarised in table 1. The value of ω is the central value among the various approximants, and the value of A^+ is that corresponding to this ω, taking into account the correlation noted above. Values of a_u^+ are obtained using (24), with

Table 1. Critical parameters of $x(y)/y$ against y^* from HTSE (present work) and, for comparison results from RPT, with a_χ^+ (HTSE) = −0.12 (Nickel and Rehr 1983).

		HTSE		RPT
y^*	0.1602	0.1608	0.1615	0.1615[a]
ω	0.79	0.78	0.78	0.79[b]
A^+	0.0087	0.013	0.024	—
a_u^+	0.21	0.25	0.32	—
a_u^+/a_χ^+	−1.75	−2.08	−2.67	−2.22

[a] Nickel and Sharpe 1979.
[b] Brézin *et al* 1973.

$C^+ = 0.199$ (Fisher and Burford 1967). Results for the critical amplitude ratio are based on the estimate $a_\chi^+ = -0.12$ for the Ising model (Nickel and Rehr (1983).

Interpolating in table 1 to *match* the results of RPT for a_u^+/a_χ^+ gives $y^* = 0.1612$, close to the RPT estimate of 0.1615. If one adopts $y^* = 0.1615$, however, one obtains an amplitude ratio a_u^+/a_χ^+ some 10% higher than that of the RPT. Thus, while these results suggest consistency between the RPT and HTSE approaches, the present uncertainty in the value of y^* prevents a definitive comparison. The need for longer χ'' series is clearly indicated.

Acknowledgments

This work was supported in part by the NSF under Grant Nos DMR-81-06151 and DMR-79-21360 (MC) and by the NSF through Grant Nos DMR-79-07238 and DMR-82-07357 (JR). We thank A Houghton and B G Nickel for discussions.

References

Aharony A and Ashers G 1980 *Phys. Rev. Lett.* **44** 782
Ahlers G 1980 *Rev. Mod. Phys.* **52** 489
Amit D J 1978 *Field Theory, The Renormalization Group, and Critical Phenomena* (New York: McGraw-Hill)
Bagnuls C and Bervillier C 1981 *Phys. Rev.* B **24** 1226
Baker G A, Nickel B G, Green M S and Meiron D J 1976 *Phys. Rev. Lett.* **36** 1351
Baker G A Jr and Kincaid J M 1979 *Phys. Rev. Lett.* **42** 1431
Bourgon A and Beysens D 1981 *Phys. Rev. Lett.* **47** 257
Brézin E, Le Guillou J C and Zinn-Justin J 1973 *Phys. Rev.* D **8** 434
—— 1976 in *Phase Transitions and Critical Phenomena* ed C Domb and M S Green (New York: Academic) vol VI
Chang M C 1980 *PhD Thesis, Brown University* (unpublished)
Chang M C and Houghton A 1980a *Phys. Rev.* B **21** 1881
—— 1980b *Phys. Rev. Lett.* **44** 785
—— 1981 *Phys. Rev.* B **23** 1473
Ferer M 1977 *Phys. Rev.* B **16** 419
Fisher M E and Burford R J 1967 *Phys. Rev.* **156** 583
McKenzie S 1979 *Can J. Phys.* **57** 1239
Nelson D R and Rudnick J 1975 *Phys. Rev. Lett.* **35** 178
Nickel B G 1980 in *Proc. Cargèse Summer Institute on Phase Transitions, Cargèse, Corsica 1980* ed M Lévy J C Guillou and J Zinn-Justin (New York: Plenum)
Nickel B G and Rehr J J 1983 to be published
Nickel B G and Sharpe B 1979 *J. Phys. A: Math. Gen.* **12** 1819
Rehr J J 1979 *J. Phys. A: Math. Gen.* **12** L179
Rogiers J, Ferer M and Scaggs E R 1979 *Phys. Rev.* B **19** 1644
Wegner F J 1972 *Phys. Rev.* B **5** 4529
Wortis M 1970 *Newport Beach Conference on Phase Transitions* (unpublished)